

A scattering approach to Casimir forces and radiative heat transfer for nanostructured surfaces out of thermal equilibrium

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We develop an exact method for computing Casimir forces and the power of radiative heat transfer between two arbitrary nanostructured surfaces out of thermal equilibrium. The method is based on a generalization of the scattering approach recently used in investigations on the Casimir effect. Analogously to the equilibrium case, we find that also out of thermal equilibrium the shape and composition of the surfaces enter only through their scattering matrices. The expressions derived provide exact results in terms of the scattering matrices of the intervening surfaces.

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I. INTRODUCTION

In recent years impressive technical advances have prompted intensive experimental and theoretical investigations of proximity phenomena, originating from quantum and thermal fluctuations of the electromagnetic field existing in the vicinity of all bodies. In general, these phenomena can be grouped in two broad classes, namely equilibrium phenomena on one side, and non-equilibrium phenomena on the other. Well-known examples of equilibrium phenomena are the Casimir effect, and the Casimir-Polder atom-wall forces [1, 2]. On the other hand, a much studied non-equilibrium phenomenon is provided by non-contact radiative heat transfer between two closely spaced bodies [3]. In recent times much interest has been devoted to the new field of Casimir and Casimir-Polder forces between macroscopic bodies and/or atoms out of thermal equilibrium. These phenomena are intensely investigated now both theoretically [4, 5] and experimentally [6]. We should also like to mention the very interesting phenomenon of non-contact quantum friction [7]. The problems of heat transfer, Casimir forces and quantum friction in a system of two plane-parallel plates at different temperatures, in relative uniform motion in a direction parallel to the plates, have also been investigated recently [8].

While the material dependence of the above phenomena has been well studied in simple planar geometries, there exists presently much interest in exploring more complicated geometries. Indeed the highly non-trivial geometry dependence of the near-field opens up the possibility of tailoring the features of the radiation field for new applications, that range from micro- and nanomachines operated by the Casimir force [9, 10], to designing the thermal emission of photonic crystals [11]. For nearly flat surfaces, the shape dependence can be studied us-

ing the so-called proximity force approximation (PFA), which amounts to averaging the plane-parallel result over the slowly varying distance between the opposing surfaces of the bodies. It is well known however that the PFA has only a limited range of validity, and it can lead to inaccurate predictions for surfaces with deep corrugations [12]. For this reason, it is widely recognized today that more accurate methods are needed to describe arbitrary geometries. For systems in equilibrium, much progress has been made recently. New powerful numerical techniques to compute the Casimir force, based on the Green's function approach and on the path-integral approach have been reported [13, 14]. Another approach that is being vigorously developed is based on the multiple scattering formalism, that was first introduced long ago [15] to study the Casimir energy for a system of perfect conductors of arbitrary shape. The scattering approach is actually closely related to the Green's function method, and we address the reader to Ref. [15] for further details on the connection between the two methods. There exist today several variants of the scattering approach, that have been developed to deal also with real materials (for a comparative review see [16]): one of the variants [17, 18] is better suited for dealing with compact objects not too close to each other, and it is based on a multipolar expansion of the e.m. field. Another variant [19] is instead better adapted to deal with planar-like structures in close proximity, and it uses a decomposition of the e.m. field into plane waves. Contrasted with these important advances in the mathematical techniques for computing equilibrium Casimir forces, the theory is much less developed for problems out of thermal equilibrium. In fact, the Casimir force out of thermal equilibrium has been investigated only in the plane-parallel case [4], while the problem of computing the near-field radiative transfer between two spheres was addressed only very recently [20], using a dyadic Green's function approach (see also the recent experiment [21]). Investigating in depth the shape-dependence of proximity effects out of thermal equilibrium is indeed very interesting, in view of

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potential applications, because out of equilibrium there exists a richness of behaviors, associated for example with resonances in the spectrum of surface excitations, that are absent at equilibrium [3, 4].

In this paper we develop an exact method for computing Casimir forces and the power of heat transfer between two arbitrary plates out of thermal equilibrium. The method is based on a generalization of the scattering approach, that has proven so successful in equilibrium Casimir problems. We consider the variant of the scattering approach [19] that is best suited for planar-like nanostructured surfaces at close separations, like those of [11, 12]. In this paper, we shall only present the derivation of the basic formulae, leaving concrete numerical applications for a successive exposition. The main new result is the demonstration that also out of thermal equilibrium, the shape and material dependence enter only through the scattering matrices of the bodies involved, analogously to what has been found for systems in thermal equilibrium. We remark that our results provide exact expressions in terms of the scattering matrices of the intervening bodies. Of course, the scattering matrix is in principle a complicated object, but there exist methods, both analytical and numerical, for computing it accurately. Many new geometries have been considered recently in Casimir investigations, and the corresponding scattering matrices have been estimated for this purpose. The formulae derived in this paper permit to consider these geometries out of thermal equilibrium.

The plan of the paper is as follows: in Sec. 2 we present the general derivation of the correlators for the e.m. field in the gap between two arbitrarily shaped plates at different temperatures. In Sec. 3 the correlators derived in Sec. 2 are used to obtain an exact expression for the Casimir force and the power of radiative heat transfer between the plates. Finally, in Sec. 4 we present our conclusions and outline directions for future work.

II. GENERAL PRINCIPLES: RYTOV'S THEORY.

We consider a geometry of the type considered in Refs. [19], i.e. a cavity consisting of two large plates at temperatures T_1 and T_2 , the whole system being in a *stationary configuration*. The shapes of their opposing surfaces can be arbitrary, apart from the assumption (usually implicit in scattering approaches to equilibrium Casimir problems [17, 18, 19]) that there must exist between them a vacuum gap of thickness $a > 0$ bounded by two parallel planes. This condition excludes from our consideration interpenetrating surfaces, like the one studied in the second of Refs. [14]. We also assume for simplicity that the plates are thick, in such a way that no radiation from outside can enter the gap.

The basic problem that we face is to determine the correlators for the fluctuating electromagnetic field existing in the empty gap between the plates. This can be done

by suitably generalizing the methods used in heat transfer studies [3], which are also at the basis of the recent out-of-equilibrium Casimir investigations [4]. Both are based on the well known Rytov's theory [22] of electromagnetic fluctuations. On the basis of this theory, the field in the gap can be interpreted as the result of multiple scatterings off the plates surfaces, of the radiation fields originating from quantum and thermal fluctuating polarizations within the plates. Importantly, the local character of the polarization fluctuations implies that the two plates radiate *independently* from each other. Therefore, in a *stationary configuration*, the radiation from either plate is the *same* as the one that would be radiated by that plate, if it were in *equilibrium* with the environment (at its own temperature), the other plate being *removed*. This physical picture permits to separate the problem of determining the fluctuating field in the gap in two separate steps. In the first step, one determines the field radiated by either plate in isolation, a problem that can be solved by using the general equilibrium formalism. The two plates are considered together only in the second step, where the intracavity field is finally determined, by taking account of the effect of multiple scatterings on the radiation fields radiated by the plates, as found in step one.

After these general remarks, we can now start our two-step computation of the fluctuating e.m. field in the gap. We let $\{x, y, z\}$ cartesian coordinates such that the vacuum gap is bounded by the planes $z = 0$ and $z = a$ respectively, with plate one (two) lying at the left (right) of the $z = 0$ ($z = a$) plane. We suppose that the lateral sizes L_x and L_y of the plates are both much larger than the separation a : $L_x, L_y \gg a$. Boundary effects being negligible, it is mathematically convenient to impose periodic boundary conditions on the fields in the (x, y) directions, on the opposite sides of the plates. When dealing with the e.m. field, it is sufficient to consider the electric field $\mathbf{E}(t, \mathbf{r})$ only, for the magnetic field $\mathbf{B}(t, \mathbf{r})$ can be obtained from $\mathbf{E}(t, \mathbf{r})$ by using Maxwell Equations. The geometry being planar-like, the electric field in the gap can always be expressed as a sum of (positive-frequency) plane-wave modes of the form

$$\mathbf{E}_{\alpha, \mathbf{k}_\perp}^{(\pm)}(t, \mathbf{r}) = 2 \operatorname{Re} [b_{\alpha, \mathbf{k}_\perp}^{(\pm)} \mathcal{E}_{\alpha, \mathbf{k}_\perp}^{(\pm)}(\omega; \mathbf{r}) e^{-i\omega t}] \quad (1)$$

where

$$\mathcal{E}_{\alpha, \mathbf{k}_\perp}^{(\pm)}(\omega; \mathbf{r}) = \mathbf{e}_{\alpha, \mathbf{k}_\perp}^{(\pm)}(\omega) e^{i\mathbf{k}^{(\pm)} \cdot \mathbf{r}}. \quad (2)$$

Here ω is the frequency, and \mathbf{k}_\perp is the projection of the wave-vector onto the (x, y) plane. Periodicity in (x, y) directions implies that the wave-vectors \mathbf{k}_\perp belong to a discrete set labelled by two integers (n_x, n_y) : $k_x = 2\pi n_x/L_x$, $k_y = 2\pi n_y/L_y$. The index $\alpha = s, p$ denotes polarization, where s and p correspond, respectively, to transverse electric and transverse magnetic polarizations. The superscripts $(+)$ and $(-)$ in Eqs. (1) and (2) refer to the direction of propagation along the z -axis, the $(+)$ and $(-)$ signs corresponding to propagation in the

positive and negative z directions, respectively. Moreover, $\mathbf{k}^{(\pm)} = \mathbf{k}_\perp \pm k_z \hat{\mathbf{z}}$, where $k_z = \sqrt{\omega^2/c^2 - k_\perp^2}$ (the square root is defined such that $\text{Re}(k_z) \geq 0$, $\text{Im}(k_z) \geq 0$), $\mathbf{e}_{s,\mathbf{k}_\perp}^{(\pm)}(\omega) = \hat{\mathbf{z}} \times \mathbf{k}_\perp$, $\mathbf{e}_{p,\mathbf{k}_\perp}^{(\pm)}(\omega) = (c/\omega) \mathbf{k}^{(\pm)} \times \mathbf{e}_{s,\mathbf{k}_\perp}^{(\pm)}$. We note that for $\omega/c > k_\perp$, when k_z is real, the modes $\mathcal{E}_{\alpha,\mathbf{k}_\perp}^{(\pm)}$ represent *propagating* waves, while for $\omega/c < k_\perp$, when k_z is imaginary, they describe *evanescent* modes. It is opportune to introduce a shortened index notation, that will prove useful in the sequel. We shall use a roman index i to denote collectively the i -th component of a vector and the position \mathbf{r} , while a greek index α will denote collectively the polarization α and the wave-vector \mathbf{k}_\perp . In this notation the i -th component of $\mathcal{E}_{\alpha,\mathbf{k}_\perp}^{(\pm)}(\omega; \mathbf{r})$ shall be denoted as $\mathcal{E}_{i\alpha}^{(\pm)}(\omega)$. Similarly, a kernel $A_{\alpha,\mathbf{k}_\perp;\alpha',\mathbf{k}'_\perp}(\omega)$ shall be denoted as $A_{\alpha,\alpha'}(\omega)$. We also set $\sum_\omega \equiv \int d\omega/(2\pi)$, $\delta_{\omega,\omega'} \equiv 2\pi\delta(\omega - \omega')$, $\sum_\alpha \equiv 1/\mathcal{A} \sum_{n_x, n_y} \sum_{\alpha'}$, and $\delta_{\alpha,\alpha'} \equiv \mathcal{A} \delta_{n_x, n'_x} \delta_{n_y, n'_y} \delta_{\alpha,\alpha'}$, where $\mathcal{A} = L_x L_y$ is the area of the plates. Finally, for any kernel $A_{\alpha,\mathbf{k}_\perp;\alpha',\mathbf{k}'_\perp}$, we define

$$\text{Tr}_\alpha A = \sum_\alpha A_{\alpha,\alpha} \equiv \frac{1}{\mathcal{A}} \sum_{n_x, n_y} \sum_\alpha A_{\alpha,\mathbf{k}_\perp;\alpha,\mathbf{k}_\perp}.$$

Having set our notations, we pass now to step one.

A. Step one: the field radiated by a single plate in thermal equilibrium

As explained above, we begin by considering each plate in isolation to determine its radiation, and we let $\mathcal{E}_i^{(A)}(\omega)$, $A = 1, 2$ the time Fourier-transform of the field radiated by plate A . The *total* radiation field $\mathcal{E}_i^{(\text{eq};A)}(\omega)$ existing, respectively, to the right of plate one and to the left of plate two, when either plate is in equilibrium at temperature T_A with the environment (the other plate being absent) can be expressed in the form:

$$\mathcal{E}_i^{(\text{eq};A)}(\omega) = \mathcal{E}_i^{(A)}(\omega) + \mathcal{E}_i^{(\text{env};A)}(\omega) + \mathcal{E}_i^{(\text{sc};A)}(\omega), \quad (3)$$

where $\mathcal{E}_i^{(\text{env};A)}(\omega)$ describes the environment radiation, including vacuum fluctuations and black-body radiation, impinging on plate A (from the right for plate one, and from the left for plate two), and $\mathcal{E}_i^{(\text{sc};A)}(\omega)$ is the corresponding scattered radiation. These fields have the expansions:

$$\mathcal{E}_i^{(A)}(\omega) = \sum_\alpha \mathcal{E}_{i\alpha}^{(\pm)}(\omega) b_\alpha^{(A)}(\omega), \quad (4)$$

$$\mathcal{E}_i^{(\text{env};A)}(\omega) = \sum_\alpha \mathcal{E}_{i\alpha}^{(\mp)}(\omega) b_\alpha^{(\text{env})}(\omega), \quad (5)$$

$$\mathcal{E}_i^{(\text{sc};A)}(\omega) = \sum_{\alpha,\alpha'} \mathcal{E}_{i\alpha}^{(\pm)}(\omega) S_{\alpha,\alpha'}^{(A)}(\omega) b_{\alpha'}^{(\text{env})}(\omega), \quad (6)$$

where, here and in Eqs. (7) and (10) below, the upper (lower) sign is for plate one (two), and $S_{\alpha\alpha'}^{(A)}$ is the scattering matrix of plate A , for radiation impinging on the right (left) surface of plate one (two). It is important to note that, for fixed plates orientations, the matrix $S_{\alpha\alpha'}^{(A)}$ depends in general on the position $\mathbf{x}^{(A)}$ of some fixed reference point $Q^{(A)}$ chosen on plate A . If $\tilde{S}_{\alpha\alpha'}^{(A)}$ is the scattering matrix of plate A relative to a coordinate system with origin at $Q^{(A)}$, then:

$$S_{\alpha\alpha'}^{(A)} = e^{-i\mathbf{k}^{(\pm)} \cdot \mathbf{x}^{(A)}} \tilde{S}_{\alpha\alpha'}^{(A)} e^{i\mathbf{k}'^{(\mp)} \cdot \mathbf{x}^{(A)}}. \quad (7)$$

The amplitudes $b_\alpha^{(\text{env})}(\omega)$ for the environment radiation in Eqs. (5) and (6) are characterized by the following non-vanishing well known correlators:

$$\langle b_\alpha^{(\text{env})}(\omega) b_{\alpha'}^{(\text{env})*}(\omega') \rangle = \frac{2\pi\omega}{c^2} F(\omega, T_A) \text{Re} \left(\frac{1}{k_z} \right) \delta_{\omega\omega'} \delta_{\alpha\alpha'} \quad (8)$$

where $F(\omega, T) = (\hbar\omega/2) \coth(\hbar\omega/(2k_B T))$, with k_B Boltzmann constant. The desired correlators for the amplitudes $b_\alpha^{(A)}(\omega)$ can now be determined by exploiting the following relation implied by the fluctuation-dissipation theorem:

$$\langle \mathcal{E}_i^{(\text{eq};A)}(\omega) \mathcal{E}_{i'}^{(\text{eq};A)*}(\omega') \rangle = \frac{2}{\omega} F(\omega, T_A) \delta_{\omega\omega'} \text{Im} G_{ii'}^{(A)}(\omega), \quad (9)$$

where $G_{ii'}^{(A)}(\omega)$ is the dyadic *retarded* Green function of plate A . In the vacuum to the right (left) of plate one (two), the Green function $G_{ii'}^{(A)}(\omega)$ can be expressed in terms of the scattering matrix $S_{\alpha\alpha'}^{(A)}$ as follows:

$$G_{ii'}^{(A)}(\omega) = G_{ii'}^{(0)}(\omega) + \frac{2\pi i \omega^2}{c^2} \sum_{\alpha\alpha'} \mathcal{E}_{i\alpha}^{(\pm)} S_{\alpha\alpha'}^{(A)} \mathcal{E}_{J(i')\alpha'}^{(\mp)} \frac{1}{k'_z}, \quad (10)$$

where $G_{ii'}^{(0)}(\omega)$ is the *retarded* Green function in free space:

$$G_{ii'}^{(0)}(\omega) = \frac{2\pi i \omega^2}{c^2} \sum_\alpha \frac{1}{k_z} \left(\theta(z - z') \mathcal{E}_{i\alpha}^{(+)} \mathcal{E}_{J(i')\alpha}^{(+)} + \theta(z' - z) \mathcal{E}_{i\alpha}^{(-)} \mathcal{E}_{J(i')\alpha}^{(-)} \right), \quad (11)$$

with $\theta(z)$ Heaviside step-function ($\theta(z) = 1$ for $z \geq 0$, $\theta(z) = 0$ for $z < 0$). Here, J denotes the *inversion* operator, whose action on space-indices is defined as $J(i) \equiv J(i, \mathbf{r}) = (i, -\mathbf{r})$. It is useful to define the action of J also on polarizations, wave-vectors and propagation directions as $J(\alpha) \equiv J(\alpha, \mathbf{k}_\perp) = (\alpha, -\mathbf{k}_\perp)$ and $J((\pm)) = (\mp)$. The following relations hold

$$\mathcal{E}_{i\alpha}^{(\pm)*} = \mathcal{E}_{J(i)\alpha}^{(\pm)} (1 + s_\alpha)/2 + \mathcal{E}_{J(i)\alpha}^{(\mp)} (1 - s_\alpha)/2, \quad (12)$$

where $s_\alpha \equiv \text{sign}(\omega^2/c^2 - k_\perp^2)$ and

$$\mathcal{E}_{J(i)J(\alpha)}^{(\pm)} = (-1)^{P(\alpha)} \mathcal{E}_{i\alpha}^{(\mp)}, \quad (13)$$

where $P(\alpha)$ is one (zero) for s (p) polarization. The reciprocity relations $G_{ii'}^{(A)}(\omega) = G_{i'i}^{(A)}(\omega)$ satisfied by the Green's function, as a consequence of microscopic reversibility, imply via Eq. (10) the following important Onsager's relations that must hold for any scattering matrix

$$S_{\alpha\alpha'}^{(A)} = \frac{k'_z}{k_z} (-1)^{P(\alpha)+P(\alpha')} S_{J(\alpha')J(\alpha)}^{(A)}. \quad (14)$$

Upon substituting the expression for $\mathcal{E}_i^{(\text{eq};A)}(\omega)$ provided by Eqs.(3-6) into the l.h.s. of Eq. (9), and after substituting the expression of the Green function Eqs. (10) into the r.h.s. of Eq. (9), by making use of Eqs. (8), (12), (13) and (14) one obtains the following expression for the non-vanishing correlators of the amplitudes $b_\alpha^{(A)}(\omega)$:

$$\langle b^{(A)}(\omega) b^{(B)\dagger}(\omega') \rangle = \delta_{AB} \frac{2\pi\omega}{c^2} F(\omega, T_A) \delta_{\omega\omega'}$$

$$\times \left(\Sigma_{-1}^{(\text{pw})} - S^{(A)} \Sigma_{-1}^{(\text{pw})} S^{(A)\dagger} + S^{(A)} \Sigma_{-1}^{(\text{ew})} - \Sigma_{-1}^{(\text{ew})} S^{(A)\dagger} \right), \quad (15)$$

where we collected the amplitudes $b_\alpha^{(A)}(\omega)$ into the (column) vector $b^{(A)}(\omega)$ and we set $\Sigma_n^{(\text{pw}/\text{ew})} = k_z^n \Pi^{(\text{pw}/\text{ew})}$, where $\Pi_{\alpha\alpha'}^{(\text{pw})} = \delta_{\alpha\alpha'} (1 + s_\alpha)/2$ and $\Pi_{\alpha\alpha'}^{(\text{ew})} = \delta_{\alpha\alpha'} (1 - s_\alpha)/2$ are the projectors onto the propagating and evanescent sectors, respectively. Eq. (15) generalizes the well known Kirchhoff's law (as can be found for example in [3]) to non-planar surfaces, and it shows that the fluctuating field radiated by plate A is fully determined by its scattering matrix $S^{(A)}$. We remark that for non-planar surfaces the matrix $S^{(A)}$ is non-diagonal, and therefore the order of the factors on the r.h.s. of Eq. (15) must be carefully respected. Now we move to step two.

B. Step two: determination of the intracavity field

Without loss of generality, the intra-cavity field can be represented as a superposition of waves of the form:

$$\mathcal{E}_{i\alpha}(\omega) = b_\alpha^{(+)}(\omega) \mathcal{E}_{i\alpha}^{(+)}(\omega) + b_\alpha^{(-)}(\omega) \mathcal{E}_{i\alpha}^{(-)}(\omega). \quad (16)$$

The intuitive physical picture of the intra-cavity field as resulting from repeated scattering off the two surfaces of the radiation field *emitted* by the surfaces of the two plates leads to the following equations for $b^{(\pm)}(\omega)$:

$$b^{(+)} = b^{(1)} + S^{(1)} b^{(-)}, \quad b^{(-)} = b^{(2)} + S^{(2)} b^{(+)}. \quad (17)$$

Equations (17) are easily solved:

$$b^{(+)} = U^{(12)} b^{(1)} + S^{(1)} U^{(21)} b^{(2)}, \quad (18)$$

$$b^{(-)} = S^{(2)} U^{(12)} b^{(1)} + U^{(21)} b^{(2)}, \quad (19)$$

where $U^{(AB)} = (1 - S^{(A)} S^{(B)})^{-1}$. Together with Eq. (15), Eqs. (18) and (19) completely determine the intracavity field. In particular, they determine the matrix $C^{(KK')}$ for the non-vanishing correlators of the intracavity field:

$$\langle b^{(K)}(\omega) b^{(K')\dagger}(\omega') \rangle = \delta_{\omega,\omega'} C^{(KK')}. \quad (20)$$

The explicit expression of $C^{(KK')}$ in terms of $S^{(1)}$ and $S^{(2)}$ can be easily obtained from Eqs. (15), (18) and (19), and it is not shown for brevity.

III. OBSERVABLES

The above results permit to evaluate the average of any observables constructed out of the intracavity field. Typically, the observables are symmetric bilinears of the electric field, of the form

$$\bar{\mathcal{O}} \equiv \sum_{ij} \int d^2 \mathbf{r}_\perp \int d^2 \mathbf{r}'_\perp E_i(t, \mathbf{r}) \mathcal{O}_{ij}(\mathbf{r}, \mathbf{r}') E_j(t, \mathbf{r}'), \quad (21)$$

where $\mathcal{O}_{ij}(\mathbf{r}, \mathbf{r}') = \mathcal{O}_{ji}(\mathbf{r}', \mathbf{r})$. Upon defining the matrix

$$\begin{aligned} \mathcal{O}_{\alpha,\alpha'}^{(KK')} &= \sum_{ij} \int d^2 \mathbf{r}_\perp \int d^2 \mathbf{r}'_\perp \mathcal{E}_{i\alpha}^{(K)*}(\omega, \mathbf{r}) \\ &\times \mathcal{O}_{ij}(\mathbf{r}, \mathbf{r}') \mathcal{E}_{j\alpha'}^{(K')}(\omega, \mathbf{r}') \end{aligned} \quad (22)$$

the statistical average of $\bar{\mathcal{O}}$ can be written as

$$\langle \bar{\mathcal{O}} \rangle = 2 \sum_{\omega > 0} \sum_{K, K'} \text{Tr}_\alpha [C^{(KK')} \mathcal{O}^{(K'K)}]. \quad (23)$$

Below we shall use this formula to determine the Casimir force and the power of heat transfer between the two plates.

A. The Casimir force out of thermal equilibrium

As our first example, we consider the (x, y) integral of the zz components of the Maxwell stress tensor T_{ij} , that provides the total Casimir force between the plates. After a simple computation, one finds:

$$\mathcal{O}^{(KK')} [T_{zz}] = \frac{c^2 k_z^2}{4\pi\omega^2} \left(\delta_{KK'} \Pi^{(\text{pw})} + \delta_{KJ(K')} \Pi^{(\text{ew})} \right). \quad (24)$$

Evaluation of Eq. (23) with $\mathcal{O}^{(KK')}$ given by Eq. (24), leads to the following representation for the *unrenormalized* Casimir force $F_z^{(0\text{neq})}$ out of thermal equilibrium:

$$F_z^{(0\text{neq})} = \sum_{\omega > 0} \frac{1}{\omega} [F(\omega, T_1) J(S^{(1)}, S^{(2)})]$$

$$+ F(\omega, T_2)J(S^{(2)}, S^{(1)})] , \quad (25)$$

where $J(S^{(A)}, S^{(B)})$ is the quantity

$$\begin{aligned} J(S^{(A)}, S^{(B)}) = & \text{Tr}_\alpha \left[U^{(AB)} \left(\Sigma_{-1}^{(\text{pw})} - S^{(A)} \Sigma_{-1}^{(\text{pw})} S^{(A)\dagger} \right. \right. \\ & + S^{(A)} \Sigma_{-1}^{(\text{ew})} - \Sigma_{-1}^{(\text{ew})} S^{(A)\dagger} \left. \right) U^{(AB)\dagger} \left(\Sigma_2^{(\text{pw})} \right. \\ & \left. \left. + S^{(B)\dagger} \Sigma_2^{(\text{pw})} S^{(B)} + \Sigma_2^{(\text{ew})} S^{(B)} + S^{(B)\dagger} \Sigma_2^{(\text{ew})} \right) \right] . \quad (26) \end{aligned}$$

After we add and subtract one half of the quantity

$$B = F(\omega, T_2)J(S^{(1)}, S^{(2)}) + F(\omega, T_1)J(S^{(2)}, S^{(1)})$$

from the expression inside the square brackets on the r.h.s. of Eq. (25), it is easily seen that Eq. (25) can be recast in the form:

$$\begin{aligned} F_z^{(0\text{neq})}(T_1, T_2) = & \frac{F_z^{(0\text{eq})}(T_1) + F_z^{(0\text{eq})}(T_2)}{2} \\ & + \Delta F_z^{(\text{neq})}(T_1, T_2) , \quad (27) \end{aligned}$$

where

$$F_z^{(0\text{eq})} = \sum_{\omega>0} \frac{1}{\omega} F(\omega, T) [J(S^{(1)}, S^{(2)}) + J(S^{(2)}, S^{(1)})] , \quad (28)$$

and

$$\begin{aligned} \Delta F_z^{(\text{neq})}(T_1, T_2) = & \sum_{\omega>0} \frac{1}{2\omega} (F(\omega, T_1) - F(\omega, T_2)) \\ & \times [J(S^{(1)}, S^{(2)}) - J(S^{(2)}, S^{(1)})] . \quad (29) \end{aligned}$$

Using the identity

$$F(\omega, T) = \hbar\omega \left[\frac{1}{2} + n(\omega, T) \right] \quad (30)$$

where

$$n(\omega, T) = \frac{1}{\exp(\hbar\omega/(k_B T)) - 1} , \quad (31)$$

Eq. (29) can be written as:

$$\begin{aligned} \Delta F_z^{(\text{neq})}(T_1, T_2) = & \frac{\hbar}{2} \sum_{\omega>0} (n(\omega, T_1) - n(\omega, T_2)) \\ & \times [J(S^{(1)}, S^{(2)}) - J(S^{(2)}, S^{(1)})] . \quad (32) \end{aligned}$$

On the other hand, upon substituting Eq. (26) into the r.h.s. of Eq. (28), after a somewhat lengthy algebraic

manipulation, it can be seen that the quantity $F_z^{(0\text{eq})}(T)$ can be further decomposed as

$$F_z^{(0\text{eq})}(T) = A^{(0)}(T) + F_z^{(\text{eq})}(T) . \quad (33)$$

Here, $A^{(0)}(T)$ denotes the divergent quantity:

$$A^{(0)}(T) = 2 \sum_{\omega>0} \frac{F(\omega, T)}{\omega} \text{Tr}_\alpha \left[k_z \Pi^{(\text{pw})} \right] . \quad (34)$$

As we see, this quantity depends neither on the material constituting the plates nor on their distance, and we neglect it altogether [24]. As to the second contribution $F_z^{(\text{eq})}$ occurring on the r.h.s. of Eq. (33), it has the expression

$$\begin{aligned} F_z^{(\text{eq})}(T) = & 2 \text{Re} \sum_{\omega \geq 0} \frac{F(\omega, T)}{\omega} \text{Tr}_\alpha \left[k_z \left(U^{(12)} S^{(1)} S^{(2)} \right. \right. \\ & \left. \left. + U^{(21)} S^{(2)} S^{(1)} \right) \right] . \quad (35) \end{aligned}$$

Recalling that, according to Eq. (7), the scattering matrices $S^{(A)}$ depend on the mutual positions of the plates, it is easy to verify that the above equilibrium force $F_z^{(\text{eq})}$ has an associated free energy $F(a, T)$ ($F_z^{(\text{eq})} = \partial F(a, T)/\partial a$) equal to:

$$F(a, T) = 2 \text{Im} \sum_{\omega \geq 0} \frac{F(\omega, T)}{\omega} \text{Tr}_\alpha \log(1 - S^{(1)} S^{(2)}) . \quad (36)$$

Eqs. (35) and (36) coincide with the equilibrium expressions, as derived within the scattering approach [19]. Putting everything together, after in Eq. (27) we remove the divergent contribution proportional to $A^{(0)}(T_1) + A^{(0)}(T_2)$, we obtain the following new *exact* expression for the renormalized Casimir force between the plates:

$$F_z^{(\text{neq})}(T_1, T_2) = \frac{F_z^{(\text{eq})}(T_1) + F_z^{(\text{eq})}(T_2)}{2} + \Delta F_z^{(\text{neq})}(T_1, T_2) . \quad (37)$$

Some comments are now in order. We note first of all that the quantities $F_z^{(\text{eq})}(T)$ and $\Delta F_z^{(\text{neq})}(T_1, T_2)$ are both *finite*. Indeed, as we said earlier, our expression for $F_z^{(\text{eq})}(T)$ coincides with the known scattering-approach expression for the equilibrium Casimir force, which has been shown to be finite in previous studies [19]. As to $\Delta F_z^{(\text{neq})}(T_1, T_2)$, it is apparent from Eq. (32) that this quantity is finite, thanks to the Boltzmann factors $n(\omega, T_i)$. We also note that our result has the same general structure as the formula derived in Refs.[4], for the simpler case of two plane-parallel plates. Analogously to that case, we indeed see from Eq. (37) that the non-equilibrium force is the sum of the average of the equilibrium forces, for the temperatures T_1 and T_2 , plus a contribution $\Delta F_z^{(\text{neq})}(T_1, T_2)$, that vanishes for $T_1 = T_2$

(see Eq. (32)). Moreover, it is interesting to observe that even for $T_1 \neq T_2$, the quantity $\Delta F_z^{(\text{neq})}$, being antisymmetric in the scattering matrices of the two plates, vanishes if the two plates have identical scattering matrices. Such a case is realized, for example, if the two plates are made of the same material and if their profiles are specularly symmetric with respect to the (x, y) plane. When this happens, the non-equilibrium Casimir force is just the average of the equilibrium forces, for the two temperatures of the plates. An analogous statement can be found in the third of Refs. [4]. We can easily verify that in the case of plane-parallel homogeneous dielectric plates our general formula Eq. (37) reproduces the result of Refs. [4]. In the flat case the scattering matrices of the plates are diagonal, and can be taken to be of the form

$$\begin{aligned} S_{\alpha\alpha'}^{(1)} &= \delta_{\alpha\alpha'} R_{\alpha}^{(1)}, \\ S_{\alpha\alpha'}^{(2)} &= \delta_{\alpha\alpha'} R_{\alpha}^{(2)} e^{2ik_z a}, \end{aligned} \quad (38)$$

where $R_{\alpha}^{(A)}$ denote the familiar Fresnel reflection coefficients. When these diagonal scattering matrices are plugged into Eq. (35), one obtains:

$$\begin{aligned} F_z^{(\text{eq})}(T) &= 4\mathcal{A} \text{Re} \sum_{\omega \geq 0} \frac{F(\omega, T)}{\omega} \sum_{\alpha} k_z \frac{R_{\alpha}^{(1)} R_{\alpha}^{(2)} e^{2ik_z a}}{1 - R_{\alpha}^{(1)} R_{\alpha}^{(2)} e^{2ik_z a}} \\ &= 4\mathcal{A} \text{Re} \sum_{\omega \geq 0} \frac{F(\omega, T)}{\omega} \sum_{\alpha} k_z \left[\frac{e^{-2ik_z a}}{R_{\alpha}^{(1)} R_{\alpha}^{(2)}} - 1 \right]^{-1}. \end{aligned} \quad (39)$$

In the limit of infinite plates, when

$$\frac{1}{\mathcal{A}} \sum_{n_x, n_y} \rightarrow \int \frac{d^2 \mathbf{k}_{\perp}}{(2\pi)^2},$$

the above formula reproduces the well known Lifshitz formula [1] for the Casimir force between two dielectric plane-parallel slabs. On the other hand, when the scattering matrices in Eq. (38) are substituted into Eq. (32), one finds:

$$\begin{aligned} \Delta F_z^{(\text{neq})}(T_1, T_2) &= \mathcal{A} \times \hbar \sum_{\omega > 0} [n(\omega, T_1) - n(\omega, T_2)] \\ &\times \sum_{\alpha} \left[\text{Re}(k_z) \frac{|R_{\alpha}^{(2)}|^2 - |R_{\alpha}^{(1)}|^2}{|1 - R_{\alpha}^{(1)} R_{\alpha}^{(2)} e^{2ik_z a}|^2} - 2 \text{Im}(k_z) e^{-2a \text{Im}(k_z)} \right. \\ &\times \left. \frac{\text{Im}(R_{\alpha}^{(1)}) \text{Re}(R_{\alpha}^{(2)}) - \text{Re}(R_{\alpha}^{(1)}) \text{Im}(R_{\alpha}^{(2)})}{|1 - R_{\alpha}^{(1)} R_{\alpha}^{(2)} e^{2ik_z a}|^2} \right]. \end{aligned} \quad (40)$$

After we substitute Eqs. (39) and (40) into Eq. (37), and upon taking the limit of infinite plates, one finds that the result coincides with the non-equilibrium Casimir force computed in Refs.[4].

B. Power of heat transfer

We consider now the total power W of heat transfer between the plates. This requires that we evaluate the statistical average of the (x, y) integral of the z -component S_z of the Poynting vector in the gap between the plates. A simple computation shows that:

$$\mathcal{O}^{(KK')} [S_z] = \frac{c^2 k_z}{4\pi\omega} (-1)^K \left(\delta_{KK'} \Pi^{(\text{pw})} + \delta_{KJ(K')} \Pi^{(\text{ew})} \right). \quad (41)$$

When this expression is plugged into Eq. (23) we obtain:

$$W = \sum_{\omega > 0} [F(\omega, T_1) H(S^{(1)}, S^{(2)}) - F(\omega, T_2) H(S^{(2)}, S^{(1)})], \quad (42)$$

where $H(S^{(A)}, S^{(B)})$ is the quantity

$$\begin{aligned} H(S^{(A)}, S^{(B)}) &= \text{Tr}_{\alpha} \left[U^{(AB)} \left(\Sigma_{-1}^{(\text{pw})} - S^{(A)} \Sigma_{-1}^{(\text{pw})} S^{(A)\dagger} \right. \right. \\ &\quad \left. \left. + S^{(A)} \Sigma_{-1}^{(\text{ew})} - \Sigma_{-1}^{(\text{ew})} S^{(A)\dagger} \right) U^{(AB)\dagger} \left(\Sigma_1^{(\text{pw})} \right. \right. \\ &\quad \left. \left. - S^{(B)\dagger} \Sigma_1^{(\text{pw})} S^{(B)} - \Sigma_1^{(\text{ew})} S^{(B)} + S^{(B)\dagger} \Sigma_1^{(\text{ew})} \right) \right]. \end{aligned} \quad (43)$$

By a lengthy computation, it is possible to verify that the quantity $H(S^{(1)}, S^{(2)})$ is symmetric under the exchange of $S^{(1)}$ and $S^{(2)}$:

$$H(S^{(1)}, S^{(2)}) = H(S^{(2)}, S^{(1)}). \quad (44)$$

By virtue of this identity, the above formula for the power of heat transfer can be rewritten as:

$$W = \hbar \sum_{\omega > 0} \omega [n(\omega, T_1) - n(\omega, T_2)] H(S^{(1)}, S^{(2)}). \quad (45)$$

We stress once again that this formula provides an *exact* expression for W in terms of the scattering matrices of the surfaces. We can consider the simple special case of two planar slabs. When the scattering matrices for two planar surfaces, given in Eq. (38), are substituted into Eq. (45), the expression for the power of heat transfer takes the following simple form:

$$\begin{aligned} W &= \mathcal{A} \times \hbar \sum_{\omega > 0} \omega [n(\omega, T_1) - n(\omega, T_2)] \\ &\times \sum_{\alpha} \left[\theta(k_z^2) \frac{(1 - |R_{\alpha}^{(1)}|^2)(1 - |R_{\alpha}^{(2)}|^2)}{|1 - R_{\alpha}^{(1)} R_{\alpha}^{(2)} e^{2ik_z a}|^2} \right. \\ &\quad \left. + \theta(-k_z^2) 4 e^{-2a \text{Im}(k_z)} \frac{\text{Im}(R_{\alpha}^{(1)}) \text{Im}(R_{\alpha}^{(2)})}{|1 - R_{\alpha}^{(1)} R_{\alpha}^{(2)} e^{2ik_z a}|^2} \right]. \end{aligned} \quad (46)$$

In the limit of large plates, the above expression coincides with the known formula for the power of heat transfer between two infinite plane-parallel dielectric slabs separated by an empty gap [3].

IV. CONCLUSIONS

In conclusion, we have developed a new *exact* method for computing Casimir forces and the power of heat transfer between two plates of arbitrary compositions and shapes at different temperatures, in vacuum. The method is based on a generalization to systems out of thermal equilibrium of the the scattering approach recently used to study the Casimir effect in non-planar geometries [17, 18, 19]. Similarly to the equilibrium case, we find that also out of thermal equilibrium the dependence on shape and material appears only through the scattering matrices of the intervening bodies. The expressions that have been obtained are exact, and lend themselves to numerical or perturbative computations once the scattering matrices for the desired geometry are evaluated. Our results provide the tool for a systematic investigation of the shape dependence of thermal proximity effects in nanostructured surfaces, that could be of interest for future applications to nanotechnology and to photonic crystals. In a successive publication [23], we

shall use the formulae derived in this paper to compute the Casimir force and the power of heat transfer between two periodic dielectric gratings, like those considered in last of Refs.[19]. The explicit form of the scattering matrices for rectangular gratings has been worked out there, on the basis of a suitable generalization of the Rayleigh expansion. At any finite order N of the Rayleigh expansion, the scattering matrices $S_{\alpha,\alpha'}$ are of the form

$$S_{\alpha,\alpha'} \equiv \hat{S}(\tilde{k}_x, k_y) \delta(\tilde{k}_x - \tilde{k}'_x) \delta(k_y - k'_y), \quad (47)$$

where $\hat{S}(\tilde{k}_x, k_y)$ is a square matrix of dimension $2(2N+1)$, \tilde{k}_x belongs to the first Brillouin zone, and k_y is unrestricted. For scattering matrices of this form, our explicit formulae for the Casimir force and the power of heat transfer can be evaluated numerically quite easily, at least for sufficiently small N .

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 - [24] In effect, the divergent quantity $A^{(0)}(T)$ includes a *finite* temperature-dependent contribution, which may give rise to a distance-independent force on the plates. The actual magnitude of the resulting constant force on either plate depends on the temperature of the environment outside the cavity, but it is independent of both the material constituting the plates, as well as of their shapes. For a detailed discussion of this point, the reader may consult

the third of Refs.[4]